Shift as a model dynamical system. Entropy and Jacobian.

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We work with the following easy dynamical system:
$X_b := \{0,\dots,b-1\}^{\mathbb{N}}$ – the sequences $(x_i)_{i\in\mathbb{N}}$ with $x_i \in \{0,\dots,b-1\}$.

$T_b : X_b \circlearrowleft$ is defined by $\left(T_b(x_i)\right)_i := x_{i+1}$ the one-sided shift.

Consider the topology on X_b generated by cylindrical sets of n-th generation: $C(x_1,\dots,x_n) := \{(y_i),\ y_i = x_i,\ i \le n\}$.
Equivalently, $T_b : [0,1] \circlearrowleft$ $x \to bx \bmod 1$, does the same with b-adic expansion, and $C(x_1,\dots,x_n)$ correspond to a b-adic interval of n-th generation. Also, $T_b : \{|z|=1\}\circlearrowleft$, $T_b(z) = z^b$, C_b – b-adic arcs.
Let $\mathcal{B}_n = \sigma\{C(x_1,\dots,x_n)\}$ – the σ-algebra generated by cylinders of n-th generation, $\mathcal{A}_n = \{C(x_1,\dots,x_n)\}$ – all n-th generation cylinders. For $A \in \mathcal{A}_n$, $B \in \mathcal{A}_m$, $AB \in \mathcal{A}_{n+m}$.

Let μ a T_b-invariant probability measure. Define
$h_n(\mu) := -\sum\limits_{A\in\mathcal{A}_n} \mu(A)\log\mu(A)$. n-entropy of μ.

Lemma. $h_{n+m}(\mu) \le h_n(\mu) + h_m(\mu)$

Corollary $\exists \lim\limits_{n\to\infty} \frac{h_n(\mu)}{n} =: h(\mu)$ – entropy of μ.

Pf of corollary. Observe that if $a_{n+m} \le a_n + a_m$, $a_n \ge 0$, then $\exists \lim\limits_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n} =: L$
Indeed, if $\frac{a_k}{k} < L + \varepsilon$, then $\frac{a_{lk}}{lk} \le L + \varepsilon$, and for $m = lk + r$,
$a_{lk+r} \le l a_k + a_r$, thus $\frac{a_{lk+r}}{lk+r} \le \frac{l a_k}{lk+r} + \frac{a_r}{lk+r} \le L + \varepsilon + \frac{\max_{r\le k} a_r}{lr} \le$
$L + 2\varepsilon$ if l is large enough ∎

Pf of lemma. Since μ is a measure, then we have for any $A \in \mathcal{A}_n$: $\mu(A) = \sum\limits_{B\in\mathcal{A}_m} \mu(AB)$.
Since μ is T invariant, $\mu(A) = \mu\left((T^{-1})^m A\right) = \sum\limits_{B\in\mathcal{A}_m} \mu(BA)$.
Since $\sum\limits_{B\in\mathcal{A}_m} \frac{\mu(AB)}{\mu(A)} = 1$, $\sum\limits_{B\in\mathcal{A}_m} \frac{\mu(AB)}{\mu(A)} \frac{\mu(B)}{\mu(AB)} = \frac{1}{\mu(A)}$,
we have, by convexity of $-\log$,
$-\log\frac{1}{\mu(A)} = \log\mu(A) \le -\sum\limits_{B\in\mathcal{A}_m} \frac{\mu(AB)}{\mu(A)} \log\frac{\mu(B)}{\mu(AB)}$ or
$\mu(A)\log\mu(A) \le -\sum \mu(AB)\log\mu(B) + \sum \mu(AB)\log\mu(AB)$
or, summing over $A \in \mathcal{A}_n$
$\sum\limits_{A\in\mathcal{A}_n} \mu(A)\log\mu(A) \le -\sum\limits_{B\in\mathcal{A}_m}\left(\sum\limits_{A\in\mathcal{A}_n} \mu(AB)\log\mu(B)\right) + \sum\sum \mu(AB)\log\mu(AB)$
or
$-h_n(\mu) \le +h_m(\mu) - h_{n+m}(\mu)$ ∎

For each cylinder of the first generation $C(i)$, the measure μ is abs. cont.
wrt $\mu_i(A) := \mu\, T^{-1}(A \cap C(i))$ (because
$\mu(A) = \sum \mu_i(A)$ by invariance) For $x \in C(i)$, define the

Jacobian of μ by $J_\mu(x) := \left(\frac{d\mu_i}{d\mu}\right)^{-1}$ (can be infinite but $J_\mu \geq 1$)

.. Measures the average expansion at x.

Thm $h_\mu(T) = \int \log J_\mu \, d\mu$.

Pf. Define $J_n(x) := \dfrac{\mu(C(x_2, \dots x_n))}{\mu(C(x_1, \dots, x_n))} = \left(\dfrac{1}{\mu(C(x_2, \dots, x_n))} \int_{C(x_2 \dots x_n)} J_\mu^{-1}(y) \, d\mu(y) \right)^{-1}$.

Then $\int J_n(x) d\mu = \sum_{C(x_1, \dots, x_n)} \mu(C(x_1, \dots x_n)) \dfrac{\mu(C(x_2, \dots, x_n))}{\mu(C(x_1, \dots x_n))} = b$.

$$\int \log J_n(x) \, d\mu = \sum_{A \in A_n} \mu(A) \log \frac{\mu(T(A))}{\mu(A)} = \sum \mu(A) \log \mu(T(A)) - \sum \mu(A) \log \mu(A)$$

$= h_n(\mu) - h_{n-1}(\mu)$.

Moreover, $\log J_n(x)$ is uniformly integrable, moreover

Claim $M(x) = \sup_n \log J_n(x) \in L^1$.

Pf of Claim $E_\lambda := \{x : M(x) > \lambda\} = \left\{x : \inf_n \dfrac{\mu(C(x_1, \dots, x_n))}{\mu(C(x_2, \dots, x_n))} < e^{-\lambda}\right\}$.

Let $C'(x)$ be the maximal cylinder containing x such that $\mu(C'(x)) < e^{-\lambda} \mu(T(C'(x)))$. $(C'(x))_{x \in E_\lambda}$ is a disjoint cover of E_λ, so

$$\mu(E_\lambda) = \sum_{C' \in \Sigma} \mu(C') < e^{-\lambda} \sum \mu(T(C')) \leq b e^{-\lambda}.$$

Thus $\int M \, d\mu = \int_0^\infty \mu(E_\lambda) \, d\lambda \leq b n$

This implies that $\log J_n \to \log J_\mu$ in L^1 and a.e.

so $\int \log J_\mu \, d\mu = \lim_{n \to \infty} \int \log J_n \, d\mu = \lim_{n \to \infty} \frac{1}{n} \int \sum_{k=0}^n \log J_k \, d\mu =$

$\lim_{n \to \infty} \frac{1}{n}(h_n(\mu) - h_0(\mu)) = h(\mu)$ ∎

Remark $\mu \to h(\mu)$ is a linear map of the space of T-invariant probability measures.

Thm (Shannon-McMillan. The meaning of entropy).

Let μ be an ergodic

μ-a.e. $\lim_{n \to \infty} -\frac{1}{n} \log \mu(C(x_1, \dots x_n)) = h(\mu)$.

Pf.

$$-\frac{1}{n} \log \mu(C(x_1, \dots, x_n)) = \frac{1}{n} \sum_{k=1}^n \log J_k(T^{n-k}(x)) =$$

$$\underbrace{\frac{1}{n} \sum_{k=1}^n \log J_\mu(T^k(x))}_{I} + \underbrace{\frac{1}{n} \sum_{k=1}^n (\log J_k - \log J_\mu)(T^{n-k}(x))}_{II}$$

By ergodic Thm, $I \to \int \log J_\mu \, d\mu = h(\mu)$.

$M_N(x) := \sup\{|\log J_k - \log J_\mu|, k \geq N\}$.

by Uniform integrability $M_N \to 0$ in $L^1(\mu)$. Choose N, so that $\int M_N < \varepsilon$.

Then $II \leq \underbrace{\frac{1}{n} \sum_{k=1}^{N-1} 2M(T^{n-k}(x))}_{II'} + \underbrace{\frac{1}{n} \sum_{k=N}^n M_N(T^{n-k}(x))}_{II''}$.

$II' = \frac{1}{n} \sum_{k=n-N}^{n-1} 2M(T^k(x))$ — tail of converging

$\frac{1}{n} \sum_{k=0}^{n-1} 2M(T^k(x))$, thus $II' \to 0$ as $n \to \infty$.

$II'' \leq \frac{1}{n} \sum_{k=1}^n M_N(T^k(x)) \xrightarrow[n \to \infty]{} \int M_N \, d\mu < \varepsilon$ ∎

Sometimes, we need to consider shift-invariant

subspaces of T_6, which correspond to T_6 invariant subsets of $[0,1]$.
Let me remind you, that for a 6×6 matrix A, $A_{ij}\in\{0,1\}$ subset $C_A \subset X_6$ (or $[0,1]$) is defined as $\{(x_i): A_{x_i x_{i+1}}=1\}$.
C_A is T_6 invariant, $T_6|_{C_A}$ is called a subshift corresponding to A.
A is called aperiodic if $\exists n: \forall i,j\ A^n_{ij}>0$, i.e. that there exists an admissible path of length n between i and j.

Def. Let X be a compact, $T: X \circlearrowleft$ - continuous. T is topologically mixing if $\forall U, V\ \exists n: \forall k \geq n\ T^k(U)\cap V \neq \emptyset$

Property. TFAE

1) $T|_{C_A}$ is topologically mixing
2) $\forall$ cylinders $C_1, C_2\ \exists n: k\geq n \quad T^k C_1 \cap C_2 \neq \emptyset$
3) A is aperiodic.

Pf. 1) $\Leftrightarrow$ 2) - Let. 2) $\Rightarrow$ 3) - consider cylinders of the first generation. 3) $\Rightarrow$ 2) Let $C_1 = C(x_1,\dots,x_\ell)$, $C_2 = C(y_1,\dots,y_m)$.
Then $\exists n: k\geq n \Rightarrow A^k_{x_\ell y_1}\geq 1 \Rightarrow \exists C(x_1,\dots x_\ell z_1 \dots z_{k-1} y_1 \dots y_m) \subset C_1 \Rightarrow T^{\ell+k} C_1 \cap C_2 \neq \emptyset$.
Topological mixing implies that for an aperiodic y, the set $\bigcup T^{-n}(y)$ is dense in C_A.